Chern Number in a Band Structure

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Words	Equations
Berry connection / vec pot	$\vec{A}(k) = i\left\langle n(k)\middle\vert\frac{\partial}{\partial \vec{k}}\middle\vert n(k)\right\rangle$
Berry Phase	$\gamma = i\int d\vec{k}\cdot\vec{A}_n(k)$
Berry Curvature	$F_n = \left\langle \vec{\nabla}_k n\middle\vert \times \middle\vert \vec{\nabla}_k n\right\rangle = \vec{\nabla}_k \times \vec{A}(k)$
Chern Number	$\nu_n = \frac{1}{2\pi}\int_{BZ} d^2k F_n \in \mathbb{Z}$
TRS with real spin	$T = -i\sigma_y K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} K$

Outline of Talk

1. Describe what Chern number means and how it is calculated.
2. Model calculation with a single Dirac cone

(a) Explain qualitatively why this is only half-quantized.
(b) Explain what this means for when a gap closes and re-opens.
(c) On hand:
    i. Calculation for state existing at edge

3. Haldane Model
    (a) Introduce Hamiltonian
        i. Full tight-binding first, without gapping terms.
        ii. Sublattice asymmetry: show it breaks inversion but not TRS
        iii. Second-nearest neighbor terms
    (b) Add periodic magnetic field
        i. Show it breaks TRS, but not inversion (necessarily)
        ii. Explain the complex hopping term qualitatively
    (c) Topological Phase Diagram
        i. Plot the gap at the different K-points as a function of M −> show why chern # changes from 0 to 1 or -1
        ii. Draw the entire phase diagram
    (d) On hand:
        i. Quantiative explanation for complex hopping term
4. QAHE in magnetic TIs
    (a) Explain system:
        i. Two surface states with finite tunneling between them (trivial mass gap)
        ii. Write down Hamiltonian
        iii. Introduce magnetic term, explain what it means
    (b) Show via same methods why it can produce finite chern # like in Haldane Model
    (c) On hand:
        i. Van Vleck Paramagnetism
        ii. Mean field theory standard
        iii. Mean field theory TI?
5. On hand for calculating quantized conductance
    (a) Anomalous velocity?
    (b) From anomalous velocity up to Hall conductivity
    (c) Influence of Dissipative Channel

1 Single Dirac Cone

$$H(k) = \sum_i d_k(k)\sigma_i \tag{1}$$

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
If $d_3 = M \neq 0$, then there will be a gap. Showing TRS is broken with a mass gap:

$$THT^{-1} = -d_x\sigma_x - d_y\sigma_y - M\sigma_z \neq H(-k) \tag{2}$$

$$H(-k) = -d_x\sigma_x - d_y\sigma_y + M\sigma_z \tag{3}$$

Generally we need $d_i(-k) = -d_i(k)$ to preserve TRS.

The normalized, orthogonal wave-functions of this two-level system are the following, incluing a helicity parameter $\alpha = \pm 1$:

$$\psi_+ = \frac{1}{\sqrt{2d(d+d_3)}} \begin{pmatrix} d_3 + d \\ (d_1 - \alpha i d_2) \end{pmatrix} \tag{4}$$

$$\psi_- = \frac{1}{\sqrt{2d(d-d_3)}} \begin{pmatrix} d_3 - d \\ (d_1 - \alpha i d_2) \end{pmatrix} \tag{5}$$

$$\psi_- = \frac{1}{\sqrt{2\sqrt{m^2+k^2}(\sqrt{m^2+k^2} - m)}} \begin{pmatrix} m - \sqrt{m^2+k^2} \\ k_x - \alpha i k_y \end{pmatrix} \tag{6}$$

$$= \frac{1}{\sqrt{B_-(k)}} \begin{pmatrix} m - \sqrt{m^2+k^2} \\ k e^{i\alpha\theta_k} \end{pmatrix} \tag{7}$$

The Berry connection is

$$\vec{A_-}(k) = i\left\langle \psi_-(k) | \vec{\nabla}_k | \psi_-(k) \right\rangle \tag{8}$$

$$= \frac{-\alpha}{2d(d-d_3)} \left(d_2 \vec{\nabla}_k d_1 - d_1 \vec{\nabla}_k d_2 \right) \tag{9}$$

$$A_x = \frac{-\alpha k_y}{2\sqrt{k^2+m^2}\left(\sqrt{k^2+m^2} - m\right)} \tag{10}$$

$$A_y = \frac{\alpha k_x}{2\sqrt{k^2+m^2}\left(\sqrt{k^2+m^2} - m\right)} \tag{11}$$

$$F_z = \frac{1}{2(k^2+m^2)^{\frac{3}{2}}} \alpha m \tag{12}$$

$$\nu = \frac{\alpha}{2\pi} \int_{-\infty}^{0} F_{xy} d^2k = \frac{\text{sign}(m\alpha)}{2} \tag{13}$$

This is for "infinite" band width of a Dirac system. However, in a real situation, finite band-width gives two gaps, each of which should roughly contribute $\pm\frac{1}{2}$ to the Chern number.

2 QAHE in TI

2.1 SS Hamiltonian in thick limit

Each surface state has a Hamiltonian

$$H_{sf}^{L} = v_F \vec{k} \times \vec{\sigma} = L i v_F \begin{pmatrix} 0 & k_- \\ -k_+ & \end{pmatrix} \tag{14}$$

where prefactor$L = \pm 1$ is for the top surface and bottom surface, and $k_\pm = k_x \pm i k_y$ Dispersion is

$$E_{sf} = v_F |k| \tag{15}$$

2.2 SS Hamiltonians with coupling

Coupling the surfaces results in a mass term.

$$H_{two} = \begin{pmatrix} 0 & iv_F k_- & m_k^* & 0 \\ -iv_F k_+ & 0 & 0 & m_k^* \\ m_k & 0 & 0 & -iv_F k_- \\ 0 & m_k & iv_F k_+ & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} t\uparrow \\ t\downarrow \\ b\uparrow \\ b\downarrow \end{pmatrix} \tag{16}$$

TRS $T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} K$ shows that $THT^{-1} = H$.

In the limit that $v_F k \ll m_k$, then $m_k \approx m_0 + m_1 k^2$ and one can rewrite the Hamiltonian in a new basis

$$H_{two} = \begin{pmatrix} m_k & iv_F k_- & 0 & 0 \\ -iv_F k_+ & -m_k & 0 & 0 \\ 0 & 0 & m_k & -iv_F k_+ \\ 0 & 0 & iv_F k_- & -m_k \end{pmatrix}, \quad \psi = \begin{pmatrix} +\uparrow \\ -\downarrow \\ +\downarrow \\ -\uparrow \end{pmatrix} \tag{17}$$

which is block diagonal yet again, with each block being an equal superposition of top and bottom surfaces, and they are both gapped:

$$E = \pm\sqrt{m_k^2 + v_F^2 k^2} \tag{18}$$

Note that this Hamiltonian is still maintains TRS, since flipping spin and complex conjugating (flipping momentum) does not change it.

2.3 Add magnetic term

A magnetic exchange term to a "bath" of localized spins inserts itself as a sort of Zeeman coupling for each spin. Therefore

$$H_M = \begin{pmatrix} gM & 0 & 0 & 0 \\ 0 & -gM & 0 & 0 \\ 0 & 0 & -gM & 0 \\ 0 & 0 & 0 & gM \end{pmatrix}, \quad \psi = \begin{pmatrix} \uparrow \\ \downarrow \\ \downarrow \\ \uparrow \end{pmatrix} \tag{19}$$

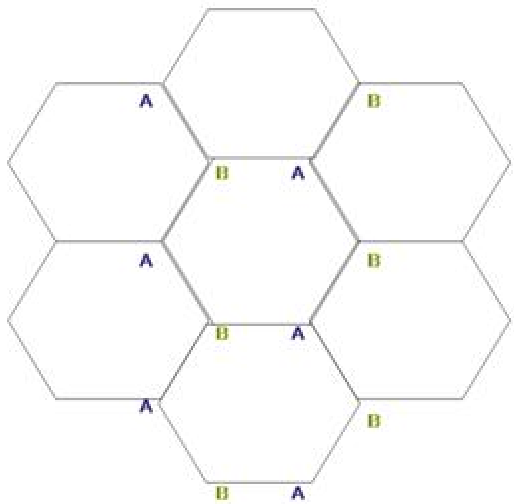
which DOES break TRS, since $THT^{-1} = -H$.

One can re-write the block-diagonal Hamiltonian as

$$H = \begin{pmatrix} h_k + (m_k + gM)\,\sigma_z & 0 \\ 0 & h_k^* + (m_k - gM)\,\sigma_z \end{pmatrix}, \quad \psi = \begin{pmatrix} +\uparrow \\ -\downarrow \\ +\downarrow \\ -\uparrow \end{pmatrix} \tag{20}$$

which shows two independent sections. When $g|M| \equiv 0$, then the system has TRS so $\nu \equiv 0$. As the exchange field $|M|$ is increased, ONE of the gaps closes when $g|M| = m_0$ and then reopens with an opposite-sign mass parameter. This means that the Chern number has changed by 1, so $\nu = \pm 1$, with the sign determined by the sign of M.

3 Haldane Model



3.1 Basic Graphene

Tight bindng for Graphene. First, lattice vectors, assume A and B are horizontal from each other: $\vec{r} = a\hat{x}$

$$\vec{r}_{a\to b} = a\hat{x} \tag{21}$$

$$= -\frac{1}{2}a\hat{x} \pm \frac{\sqrt{3}}{2}a\hat{y} \tag{22}$$

$$\vec{r}_{b\to a} = -a\hat{x} \tag{23}$$

$$= \frac{1}{2}a\hat{x} \pm \frac{\sqrt{3}}{2}a\hat{y} \tag{24}$$

But either way:

$$H = t_1 \sum_{\langle i,j\rangle} e^{i\vec{k}\cdot\vec{r}_{ij}} c_i^\dagger c_j + \sum_{i=a,b} \epsilon_i c_i^\dagger c_i \tag{25}$$

Find the zero-energy spots when $m = 0$. We know it is at the K point, which we can find by setting $k_x = 0$:

$$1 + 4\cos^2(\frac{\sqrt{3}}{2}ak_y) + 4\cos(\frac{\sqrt{3}}{2}ak_y)\cos(\frac{1}{2}ak_x) = 0 \tag{26}$$

$$1 + 4\cos^2(\frac{\sqrt{3}}{2}ak_y) + 4\cos(\frac{\sqrt{3}}{2}ak_y) = 0 \tag{27}$$

$$\left(1 + 2\cos(\frac{\sqrt{3}}{2}ak_y)\right)^2 = 0 \tag{28}$$

$$\cos(\frac{\sqrt{3}}{2}ak_y) = -\frac{1}{2} \tag{29}$$

$$\frac{\sqrt{3}}{2}ak_y = \pm\frac{2\pi}{3} \tag{30}$$

$$K_\pm = \pm\frac{4\pi}{3\sqrt{3}}\frac{1}{a}\hat{k_y} \tag{31}$$

Now simplify the Hamiltonian:

$$
\begin{aligned}
H &= t_1 \begin{pmatrix} 0 & e^{-iak_x} + e^{i\frac{1}{2}ak_x}\left(e^{i\frac{\sqrt{3}}{2}ak_y} + e^{-i\frac{\sqrt{3}}{2}ak_y}\right) \\ e^{iak_x} + e^{-i\frac{1}{2}ak_x}\left(e^{i\frac{\sqrt{3}}{2}ak_y} + e^{-i\frac{\sqrt{3}}{2}ak_y}\right) & 0 \end{pmatrix} && (32) \\
&= t_1 \begin{pmatrix} 0 & e^{-ia\delta_x} + e^{i\frac{1}{2}a\delta_x}\left(e^{\pm\frac{2\pi}{3}i+\frac{\sqrt{3}}{2}ia\delta_y} + e^{\mp\frac{2\pi}{3}i-\frac{\sqrt{3}}{2}ia\delta_y}\right) \\ e^{ia\delta_x} + e^{-i\frac{1}{2}a\delta_x}\left(e^{\pm\frac{2\pi}{3}i+\frac{\sqrt{3}}{2}ia\delta_y} + e^{\mp\frac{2\pi}{3}i-\frac{\sqrt{3}}{2}ia\delta_y}\right) & 0 \end{pmatrix} && (33) \\
&= t_1 \begin{pmatrix} 0 & e^{-ia\delta_x} + e^{i\frac{1}{2}a\delta_x}\left(e^{\pm\frac{2\pi}{3}i}e^{\frac{\sqrt{3}}{2}ia\delta_y} + e^{\mp\frac{2\pi}{3}i}e^{-\frac{\sqrt{3}}{2}ia\delta_y}\right) \\ e^{ia\delta_x} + e^{-i\frac{1}{2}a\delta_x}\left(e^{\pm\frac{2\pi}{3}i}e^{\frac{\sqrt{3}}{2}ia\delta_y} + e^{\mp\frac{2\pi}{3}i}e^{-\frac{\sqrt{3}}{2}ia\delta_y}\right) & 0 \end{pmatrix} && (34) \\
&\approx t_1 \begin{pmatrix} 0 & 1 - ia\delta_x + \left(1 + i\frac{a}{2}\delta_x\right)\left(-1 + \alpha\frac{3}{2}a\delta_y\right) \\ 1 + ia\delta_x + \left(1 - i\frac{a}{2}\delta_x\right)\left(-1 + \alpha\frac{3}{2}a\delta_y\right) & 0 \end{pmatrix} && (35) \\
&\approx t_1 \begin{pmatrix} 0 & 1 - ia\delta_x + \left(-1 + \alpha\frac{3}{2}a\delta_y - i\frac{a}{2}\delta_x\right) \\ 1 + ia\delta_x + \left(-1 + \alpha\frac{3}{2}a\delta_y + i\frac{a}{2}\delta_x\right) & 0 \end{pmatrix} && (36) \\
&\approx \frac{3}{2}at_1 \begin{pmatrix} 0 & \alpha\delta_y - i\delta_x \\ \alpha\delta_y + i\delta_x & 0 \end{pmatrix} && (37)
\end{aligned}
$$

where $\frac{3}{a}at_1$ is the fermi velocity v_F up to a factor of $\hbar$. Note that $\alpha = \pm 1$ for K/K'. This leads to the familiar dispersion relation

$$E = \pm\hbar v_F|\delta| \tag{38}$$

3.2 Symmetries

3.2.1 Inversion

$\vec{r} \to -\vec{r}$. This swaps the sublattices. It clearly does not affect the n.n. off-diagonal components, but it does affect the on-site energies. Breaking inversion opens a band gap.

3.2.2 TRS

$\hat{T} = iK\sigma_y$, so $T^2 = -1$. The Hamiltonian, as is, has TRS, i.e.

$$THT^{-1} = H \tag{39}$$

A term in the Hamiltonian that breaks TRS may open up a band gap.

3.3 Second Nearest Neighbor

First, need the vectors:

$$
\begin{aligned}
\vec{r}_{a\to a} &= \pm\sqrt{3}a\hat{y} && (40) \\
&= \pm\frac{3}{2}a\hat{x} \pm \frac{\sqrt{3}}{2}a\hat{y} && (41) \\
\vec{r}_{b\to b} &= \pm\sqrt{3}a\hat{y} && (42) \\
&= \pm\frac{3}{2}a\hat{x} \pm \frac{\sqrt{3}}{2}a\hat{y} && (43)
\end{aligned}
$$

The perturbation to the model is

$$H_2 = t_2 \sum_{\langle\langle i,j\rangle\rangle} e^{i\vec{k}\cdot\vec{r}_{ij}} c_i^\dagger c_j \tag{44}$$

which is a diagonal term. As written, this term does not break either symmetry of the lattice, and so does not open a gap. It contributes the following (this is at K^+):

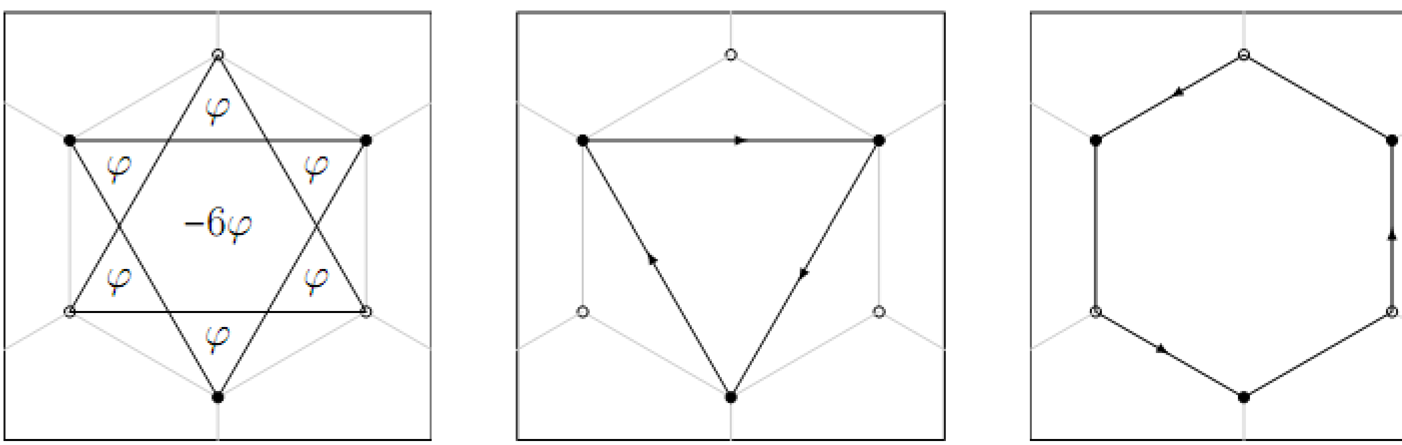
$$\begin{aligned} H_{nnn} &= 2t_2\left(\cos\left(\sqrt{3}ak_y\right)+\cos\left(\frac{3}{2}ak_x-\frac{\sqrt{3}}{2}k_y\right)+\cos\left(\frac{3}{2}ak_x+\frac{\sqrt{3}}{2}k_y\right)\right) \quad (45)\\ &= 2t_2\left(\cos\left(\sqrt{3}a\left(\frac{4\pi}{3\sqrt{3}}\frac{1}{a}+\delta_y\right)\right)+\cos\left(\frac{3}{2}a\delta_x-\frac{\sqrt{3}}{2}\left(\frac{4\pi}{3\sqrt{3}}\frac{1}{a}+\delta_y\right)\right)+\cos\left(\frac{3}{2}a\delta_x+\frac{\sqrt{3}}{2}\left(\frac{4\pi}{3\sqrt{3}}\frac{1}{a}+\delta_y\right)\right)\right) \quad (46)\\ &= 2t_2\left(\cos\left(\frac{4\pi}{3}+\sqrt{3}a\delta_y\right)+\cos\left(\frac{3}{2}a\delta_x-\frac{2\pi}{3}-\frac{\sqrt{3}}{2}a\delta_y\right)+\cos\left(\frac{3}{2}a\delta_x+\frac{2\pi}{3}+\frac{\sqrt{3}}{2}a\delta_y\right)\right) \quad (47)\end{aligned}$$

And the key is that at $\delta_{x,y}=0$:

$$\begin{aligned} H_{nnn}(0,0) &= 2t_2\left(\cos\left(\frac{4\pi}{3}\right)+\cos\left(-\frac{2\pi}{3}\right)+\cos\left(\frac{2\pi}{3}\right)\right) \quad (48)\\ &= 2t_2\left(\cos\left(\frac{4\pi}{3}\right)+2\cos\left(\frac{2\pi}{3}\right)\right) \quad (49)\\ &= -3t_2 \quad (50)\end{aligned}$$

which is just a chemical potential shift ($K^\pm$ is the same since any $\pm$ happens in front of a δ_y) and has no consequence. When $\delta_{x,y}\neq 0$, then this term renormalizes the Fermi velocity, and will be different for electrons and holes, but doesn't make any fundamental change in the band structure.

3.4 Haldane Term



Now introduce a periodic magnetic flux into the unit cell and choose a gauge so that only the nnn hoppings develop a phase.

$$H_2 = t_2\sum_{\langle\langle i,j\rangle\rangle} e^{i\phi_{ij}}e^{i\vec{k}\cdot\vec{r}_{ij}}c_i^\dagger c_j \quad (51)$$

Be careful about minus signs!

$$\begin{aligned} H_{nnn,a} &= 2t_2\left(\cos\left(\sqrt{3}ak_y+\phi\right)+\cos\left(\frac{3}{2}ak_x-\frac{\sqrt{3}}{2}k_y+\phi\right)+\cos\left(\frac{3}{2}ak_x+\frac{\sqrt{3}}{2}k_y+\phi\right)\right) \quad (52)\\ H_{nnn,b} &= 2t_2\left(\cos\left(\sqrt{3}ak_y-\phi\right)+\cos\left(\frac{3}{2}ak_x-\frac{\sqrt{3}}{2}k_y-\phi\right)+\cos\left(\frac{3}{2}ak_x+\frac{\sqrt{3}}{2}k_y-\phi\right)\right) \quad (53)\end{aligned}$$

Recall that

$$\cos(a\pm b) = \cos a\cos b \mp \sin a\sin b \quad (54)$$

$$H_{nnn,a} = 2t_2\cos\phi\left(\cos\left(\sqrt{3}ak_y\right) + \cos\left(\frac{3}{2}ak_x - \frac{\sqrt{3}}{2}k_y\right) + \cos\left(\frac{3}{2}ak_x + \frac{\sqrt{3}}{2}k_y\right)\right) \quad (55)$$

$$-2t_2\sin\phi\left(\sin\left(\sqrt{3}ak_y\right) + \sin\left(\frac{3}{2}ak_x - \frac{\sqrt{3}}{2}k_y\right) + \sin\left(\frac{3}{2}ak_x + \frac{\sqrt{3}}{2}k_y\right)\right) \quad (56)$$

$$H_{nnn,b} = 2t_2\cos\phi\left(\cos\left(\sqrt{3}ak_y\right) + \cos\left(\frac{3}{2}ak_x - \frac{\sqrt{3}}{2}k_y\right) + \cos\left(\frac{3}{2}ak_x + \frac{\sqrt{3}}{2}k_y\right)\right) \quad (57)$$

$$+2t_2\sin\phi\left(\sin\left(\sqrt{3}ak_y\right) + \sin\left(\frac{3}{2}ak_x - \frac{\sqrt{3}}{2}k_y\right) + \sin\left(\frac{3}{2}ak_x + \frac{\sqrt{3}}{2}k_y\right)\right) \quad (58)$$

So the mass-inducing terms are

$$H_{nnn}(0,0,a) = -2t_2\sin\phi\left(\sin\left(\sqrt{3}ak_y\right) + \sin\left(\frac{3}{2}ak_x - \frac{\sqrt{3}}{2}k_y\right) + \sin\left(\frac{3}{2}ak_x + \frac{\sqrt{3}}{2}k_y\right)\right) \quad (59)$$

$$H_{nnn}(0,0,b) = 2t_2\sin\phi\left(\sin\left(\sqrt{3}ak_y\right) + \sin\left(\frac{3}{2}ak_x - \frac{\sqrt{3}}{2}k_y\right) + \sin\left(\frac{3}{2}ak_x + \frac{\sqrt{3}}{2}k_y\right)\right) \quad (60)$$

So

$$H_a - H_b = 4t_2\sin\phi\left(\sin\left(\frac{4\pi}{3}\right) + \sin\left(\frac{3}{2}a\delta_x - \frac{\sqrt{3}}{2}a\delta_y - \frac{2\pi}{3}\right) + \sin\left(\frac{3}{2}a\delta_x + \frac{\sqrt{3}}{2}a\delta_y + \frac{2\pi}{3}\right)\right) \quad (61)$$

$$= 4t_2\sin\left(\frac{4\pi}{3}\right)\sin(\phi) \quad (62)$$

$$= -2\sqrt{3}t_2\sin\phi \quad (63)$$

Note that the sign of this term in the Hamiltonian is DEPENDENT on the valley since each valley has a different helicity.

$$@K \rightarrow M - t_2\sin\phi \quad (64)$$

$$@K' \rightarrow M + t_2\sin\phi \quad (65)$$

$$@K \rightarrow \nu = \frac{\text{sign}\left(M - t_2\sin\phi\right)}{2} \quad (66)$$

$$@K' \rightarrow \nu = \frac{-\text{sign}\left(M + t_2\sin\phi\right)}{2} \quad (67)$$

See that when M goes from $+\infty$ to $< t_2\sin\phi$, the gap at K' stays open always and at K it closes and reopens, with the parameter changing sign. This means the total Chern # much change by an integer 1, and we know that at $M \gg t_2\sin\phi$ it is trivial due to the system appearing like an atomic system.

4 Supplementary Calculations

4.1 Edge Modes at Mass Inversion Interface

Hamiltonian of gapped Dirac system with mass that depends on the y coordinate.

$$H(y) = -i\frac{\partial}{\partial x}\sigma_x - i\frac{\partial}{\partial y}\sigma_y + m(y)\sigma_z \quad (68)$$

and look for a separable solution

$$\psi(x,y) = \phi_x(x)\phi_y(y) \quad (69)$$

Now, $\phi_x = e^{ik_X x}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, whereas

$$\phi_y = e^{-\int_0^y m(y')dy'} \tag{70}$$

Plugging in $\phi_y\phi_x$, one gets a function of $\phi_x$and m to find the exact spinor for ϕ_x. Then you can solve for the plane-wave part which gives a chiral mode.

4.2 Phase From Magnetic Flux

Phase acquired from a path with a vector potential:

$$\phi = \frac{e}{h}\int \vec{A}\cdot d\vec{r} \tag{71}$$

Which is Gauge dependent if not on a closed loop:

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi \rightarrow \phi' \neq \phi \tag{72}$$

But if you have a closed loop, then you have a fixed quantity:

$$\varphi = \frac{e}{h}\oint \vec{A}\cdot d\vec{r} = \frac{e}{h}\iint \vec{B}\cdot d\vec{s} \tag{73}$$

And the complex phase around the loop is related to this phase. You can put this phase on whatever hopping term you like, due to the flexibility of gauge transformations.

4.3 Adiabatic Continuity

General Hamiltonian and solution to id:

$$\hat{H}(t)\psi_n(t) = E_n(t)\psi_n(t) \tag{74}$$

$$\Psi(t) = \sum_n c_n(t)\psi_n(t)e^{i\theta_n(t)} \tag{75}$$

$$\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t')dt' \tag{76}$$

Subtitute Ψ into $\hat{H}(t)$ and you get

$$i\hbar\sum_n\left(\dot{c}_n\psi_n + c_n\dot{\psi}_n + ic_n\psi_n\dot{\theta}_n\right)e^{i\theta_n} = \sum_n c_n\hat{H}\psi_n e^{i\theta_n} \tag{77}$$

but $\dot{\theta}_n = -\frac{E}{\hbar}$ so

$$\sum_n \dot{c}_n\psi_n e^{i\theta_n} = -\sum_n c_n\dot{\psi}_n e^{i\theta_n} \tag{78}$$

now inner product with an arbitrary eigenfunction ψ_m and you get

$$\dot{c}_m(t) = -\sum_n c_n\left\langle \psi_m|\dot{\psi}_n\right\rangle e^{i(\theta_n-\theta_m)} \tag{79}$$

then calculating the differentiated inner product you get the exact form

$$\dot{c}_m(t) = -c_m\left\langle \psi_m|\dot{\psi}_m\right\rangle - \sum_{n\neq m} c_n\frac{\left\langle \psi_m|\dot{\hat{H}}|\psi_n\right\rangle}{E_n - E_m}e^{i(\theta_n-\theta_m)} \tag{80}$$

but if $\frac{\partial}{\partial t}\hat{H} \ll 1$, then the second term drops out. This means that

$$c_m(t) = c_m(0)e^{-\int_0^t\langle\psi_m|\dot{\psi}_m\rangle dt'} = c_m(0)e^{i\gamma_m(t)} \tag{81}$$

so that there is a dynamical phase factor known as the geometrical phase, but the amplitudes remain constant. Thus, a particle remains in the n^{th} eigenstate at all times if the Hamiltonian varies slowly.

4.4 Van Vleck Paramagnetism

Assume an insulator with no net magnetism, i.e.

$$\mu_z \cdot B = 0 \tag{82}$$

so that there is no first-order correction to the system. But then let's say that

$$\langle c|S_z|v\rangle \neq 0 \tag{83}$$

This means that with perturbation theory in a field $\mu B \ll E_G$, one gets a wave-function

$$\psi_v' = \psi_v + \frac{B}{E_G}\langle c|S_z|v\rangle\,\psi_c \tag{84}$$

And now we have a moment

$$\langle 0'|S_z|0'\rangle \approx \frac{2B}{E_G}\left|\langle c|S_z|v\rangle\right|^2 \tag{85}$$

So that in the presence of an applied field the magnetization is

$$\frac{M}{N} = \frac{2B}{E_G}\left|\langle c|S_z|v\rangle\right|^2 \tag{86}$$

And for a band insulator with more than one quantum number k then

$$\chi = 2\sum_{(c,v),k}\frac{\left|\langle c|S_z|v\rangle\right|^2}{E_{c,k}-E_{v,k}} \tag{87}$$

In typical systems, these terms are very small. What happens in a TI? Well, there is a point in the TI band structure where the gap comes close to closing and at the same time this causes a mazimization of the second order effects.

4.5 Mean Field Theory

4.5.1 Normal FM

$$H = -J\sum_{\langle i,j\rangle} s_i s_j - h\sum_i s_i \tag{88}$$

so define the mean field $m_i = \langle s_i\rangle$ and rewrite

$$H = -J\sum_{\langle i,j\rangle}(m_i+\delta s_i)(m_j+\delta s_j) - h\sum_i s_i \tag{89}$$

$$H \approx H^{MF} \tag{90}$$

$$H^{MF} = -J\sum_{\langle i,j\rangle}(m_i m_j + m_i\delta s_j + m_j \delta s_i) - h\sum_i s_i \tag{91}$$

$$= -J\sum_{\langle i,j\rangle}\left(m^2 + 2m(s_i - m)\right) - h\sum_i s_i \tag{92}$$

and $\sum_{\langle i,j\rangle} = \frac{1}{2}\sum_i\sum_{j\in nn(i)}$ and thus $\sum_{j\in nn(i)} = x$ which is the number of nearest-neighbor spins. Therefore

$$H^{MF} = -\frac{1}{2}Jm^2Nx - (h+mJx)\sum_i s_i \tag{93}$$

This gives us a partition function

$$Z = e^{-\frac{Jm^2Nx}{2\beta}}\left[2\cosh\left(\frac{h+mJx}{k_BT}\right)\right]^N \tag{94}$$

which gives a Free Energy

$$F = -k_B T \ln Z \sim N k_B T \ln\left[2\cosh\left(\frac{h + mJx}{k_B T}\right)\right] \tag{95}$$

$$M = -\frac{\partial F}{\partial H} \sim \tanh\left(\frac{h + mJx}{k_B T}\right) \tag{96}$$

4.6 Lagrangian for wave-packet

Marder, section 16.4, page 425.

$\mathcal{L} = \langle w | i\hbar \frac{\partial}{\partial t} | w \rangle - \left\langle w | \vec{H} - eV | w \right\rangle$, assuming no magnetic fields so $\mathcal{H} = \frac{p^2}{2m} + U(r)$ and $|w\rangle = \frac{1}{\sqrt{N}} \sum_k w_{kk_c} e^{-i\vec{k}\cdot\vec{r}_c} \psi_k$ and $w_{kk_c} = e^{i(k-k_c)\cdot\vec{A}_{berry}}$ and $\mathcal{H}\psi_k = \mathcal{E}_k \psi_k$.

The Lagrangian after math simplifies to

$$\mathcal{L} = \hbar\vec{k}_c \cdot \dot{\vec{r}}_c + \hbar\dot{\vec{k}}_c \cdot \vec{A}_{berry} - \mathcal{E}_{k_c} - eV(\vec{r}_c) \tag{97}$$

$$\frac{\partial \mathcal{L}}{\partial \vec{r}_c} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_c} \text{ and } \frac{\partial \mathcal{L}}{\partial \vec{k}_c} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{k}}_c} \tag{98}$$

$$-e\vec{E} = \frac{d}{dt}\left(\hbar\vec{k}_c\right) \tag{99}$$

$$\hbar\dot{\vec{r}}_c + \frac{\partial \mathcal{E}_k}{\partial k_c} = \frac{d}{dt}\frac{\partial}{\partial \dot{\vec{k}}_c}\left(\hbar\dot{\vec{k}}_c \cdot \vec{A}_{berry}\right) \tag{100}$$

and eventually with a bunch of vector identities, etc.

$$\dot{\vec{k}} = -\frac{e}{\hbar}\vec{E} \tag{101}$$

$$\dot{\vec{r}} = \frac{1}{\hbar}\frac{\partial \mathcal{E}_k}{\partial k} - \dot{\vec{k}} \times \vec{F} \tag{102}$$

4.7 Anomalous Velocity

Semiclassical calculations (or Kubo formula, or other means), give the following two equations of motion for a wave-packet when the external magnetic field is zero:

$$\dot{\vec{k}} = -\frac{e}{\hbar}\vec{E} \tag{103}$$

$$\dot{\vec{r}} = \frac{1}{\hbar}\frac{\partial \mathcal{E}_k}{\partial k} - \dot{\vec{k}} \times \vec{F} \tag{104}$$

while if the system is in a band gap then $\int_{BZ} \frac{\partial E(k)}{\partial k} \equiv 0$, e.g. when integrating over completely filled bands. As such, I pre-emptively remove that term from the calculation (reader can verify for him/herself that this is ok). So, substituting, we get

$$\dot{\vec{r}} = v(k) = \frac{e}{\hbar}\vec{E} \times \vec{F} \tag{105}$$

For a 2D system, $\vec{F} = F_z$. To calculate the conductivity one needs

$$\vec{j} = \hat{\sigma}\vec{E} \tag{106}$$

so

$$j_y = \sigma_{xy} E_x \tag{107}$$

then, noting that the distribution function g is roughly the Fermi function $g \sim f$ in this context.

$$
\begin{aligned}
j_y &= e \iint \frac{d^2k}{4\pi^2} \vec{v}(k) g & (108)\\
&= \frac{e^2}{h} \frac{1}{2\pi} \iint d^2k \vec{E} \times \vec{F} f & (109)\\
&= \frac{e^2}{h} \frac{1}{2\pi} E_x \iint d^2k F_z & (110)\\
\sigma_{xy} &= \frac{dj_y}{dE_x} & (111)\\
&= \frac{e^2}{h} \frac{1}{2\pi} \iint d^2k F_z & (112)\\
&= \frac{e^2}{h} \nu & (113)\\
\nu &= \frac{1}{2\pi} \iint d^2k F_z \in \mathbb{Z} & (114)
\end{aligned}
$$

Normally,

$$
g \approx f - \tau e \frac{\partial f}{\partial \mu} \vec{v}_{\vec{k}} \cdot \vec{E} \qquad (115)
$$

and when you take the derivative $\frac{dj_\beta}{dE_\alpha}$ the first term drops out since there is no electric field there. However, when inside a gap and with finite Berry curvature, $\vec{v}(k)\frac{\partial f}{\partial \mu} = 0$ and the only remaining term is the one with the Berry curvature multiplying the general Fermi function.

4.8 QHE and dissipative channels

$$
\begin{aligned}
\sigma_{xy} &= \sigma_{QH} + \sigma_{dis} = \sigma_{QH} & (116)\\
\sigma_{xx} &= \sigma_{QH} + \sigma_{dis} \approx \sigma_{dis} & (117)
\end{aligned}
$$

so

$$
\begin{aligned}
\sigma &= \begin{pmatrix} \sigma_d & \sigma_Q \\ -\sigma_Q & \sigma_d \end{pmatrix} & (118)\\
\rho &= \frac{1}{\sigma_d^2 + \sigma_Q^2} \begin{pmatrix} \sigma_d & -\sigma_Q \\ \sigma_Q & \sigma_d \end{pmatrix} & (119)
\end{aligned}
$$

so

$$
\begin{aligned}
\rho_{xx} &= \frac{\sigma_d}{\sigma_d^2 + \sigma_Q^2} \approx \frac{h}{e^2} \frac{\xi}{\xi^2 + 1} & (120)\\
\rho_{xy} &= \frac{\sigma_Q}{\sigma_d^2 + \sigma_Q^2} \approx \frac{h}{e^2} \frac{1}{\xi^2 + 1} & (121)
\end{aligned}
$$

So as the dissipative channel's conductance goes away, the dissipating resistance goes to zero, and the Hall resistance approaches the quantized value.